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Note on a parametrically excited, trapped cross-wave

By JOHN W. MILES

Institute of Geophysics and Planetary Physics, University of California, San Diego, La Jolla, California 92093

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The cubic Schrödinger equation, which governs the complex amplitude of a cross-wave that is subharmonically excited by a symmetric wavemaker in a deep wave tank (Jones 1984), is solved for prescribed boundary conditions at the wavemaker and a null condition at infinity. This solution (which is a limiting case of a one-parameter family of cnoidal waves) typically describes a trapped wave that decays exponentially in a semi-infinite tank, although it may be fitted to a tank of finite length for a particular combination of parameters. It is closely related to the solution for trapped waves excited through vertical oscillation of a long channel (Wu, Keolian & Rudnick 1984).

1. Introduction

Jones (1984) considers the generation of cross-waves in a deep wave tank that is driven by a wavemaker with the prescribed motion

$$x = af(z)\sin 2\sigma t \quad (z < 0). \tag{1}$$

He posits the velocity potential for a cross-wave with n transverse nodes in the form (I have restored dimensions except as noted)

$$\psi = \frac{ga}{\sigma}[C(X,\tau)\cos\sigma t + D(X,\tau)\sin\sigma t]\cos\left(\frac{n\pi y}{b}\right)\exp(kz), \tag{2}$$

where C and D are dimensionless, slowly varying amplitudes,

$$X = \epsilon kx, \quad \tau = \epsilon^2\sigma t, \quad \epsilon = ka, \quad k = \frac{\sigma^2}{g}, \tag{3a,b,c,d}$$

$n = 1, 2, \dots$ and b is the breadth of the tank. He then finds that C and D satisfy a pair of evolution equations, his (38), that may be combined to obtain the cubic Schrödinger equation

$$\mathrm{i}\frac{\partial F}{\partial \tau} + \frac{1}{4}\frac{\partial^2 F}{\partial X^2} - \tfrac{1}{2}JF + \tfrac{1}{8}|F|^2 F = 0, \tag{4}$$

where

$$F \equiv C + \mathrm{i}D,$$

$$J = \lambda + 0.404A^2, \quad A = 4k\int_{-\infty}^{0} f(z)\,\mathrm{e}^{4kz}\,\mathrm{d}z, \tag{5a,b†}$$

† I have corrected a sign error in the last term in Jones's (35a), which leads to a change of the sign of λ in his (36a), (37) and (39). Jones (private communication) agrees with these changes.

$$\lambda = \frac{1}{\epsilon^2}\left(\frac{n\pi}{kb}-1\right) = \frac{\sigma_n^2-\sigma^2}{\epsilon^2\sigma^2}, \tag{6}$$

and $\sigma_n = (n\pi g/b)^{\frac{1}{2}}$ is the resonant (or *cutoff*) frequency of the cross-wave. His boundary conditions (34c) may be combined to obtain

$$\frac{\partial F}{\partial X} = \mathrm{i}LF^* \quad (X=0), \tag{7}$$

where F^* is the complex conjugate of F, and

$$L = \int_{-\infty}^{0} [4kf(z)+f'(z)\,\mathrm{e}^{2kz}]\,\mathrm{d}z - 2f(0). \tag{8}$$

F also must satisfy appropriate conditions at some downstream station.

On examining the corresponding problem for a tank of finite depth d, for which $\exp(kz)$ is replaced by $\cosh[k(z+d)]/\cosh kd$ in (2) and the dispersion relation (3d) is replaced by $k\tanh kd = \sigma^2/g$, I find that (cf. Miles 1984) the terms $\partial^2F/\partial X^2$ and $|F|^2F$ in (4) must be multiplied by functions of $\kappa = \pi d/b$ that differ from 1 by exponentially small terms as $\kappa\uparrow\infty$ and are within 5 % of 1 for $\kappa > 2.5$. Similar changes must be made in the parameters A and L.

Representative values of the parameters A and L are provided by Barnard & Pritchard's (1972) experiments,† for which $b = 30.6$ cm, $n = 2/3$ (2 *or* 3), $d = 16.4/16.1$ cm, $2\sigma = 28.30/34.86$ rad/s, and

$$f(z) = 1+\frac{z}{d} \quad (-d \leqslant z \leqslant 0). \tag{9}$$

The corresponding value of kd, as given by (3d) with $g = 981$ cm/s^2, is 3.34/4.99, for which the deep-water approximation is amply justified. The substitution of (9) into (5b) and (8), followed by the neglect of $O[\exp(-2kd)]$, yields

$$A = 1-(4kd)^{-1}, \quad L = (2kd)^{-1}(2kd-1)^2, \tag{10a,b}$$

which reduce to 0.93/0.95 and 4.83/8.08 respectively for $kd = 3.34/4.99$. The corresponding value of $n\pi/kb$ is 1.006/0.994, which implies that the excitation is just below/above the cutoff frequency; however, the difference $\sigma-\sigma_n$ is of the same order as (the neglected) capillary and viscous effects (and perhaps also the uncertainties in the dimensions).

2. Trapped solutions

Jones considers numerical solutions of (4), (7) and the condition $\partial F/\partial X = 0$ at $X = 1$ for $J = 0$, but appears to have overlooked the existence of trapped solutions that may be derived from the known solitary-wave solution of (4) for $J > 0$ and are at least qualitatively consistent with observations of cross-waves in appropriate parametric domains. The boundary condition (7) sharply constrains the general solitary-wave solution of (4) (Whitham 1974), but it does admit solutions of the form

$$F(X) = (1\pm\mathrm{i})\,C(X) \tag{11}$$

† Barnard & Pritchard describe their tank as 'long' (their figure 3 implies that its length exceeds 4b), but then state that its length is 2.7 cm; this should be 2.7 m (Pritchard, private communication).

(i.e. $D = \pm C$). Substituting (11) into (4) and invoking (7) and $C(\infty) = 0$, I find that the only non-trivial solutions are given by

$$C = 2J^{\frac{1}{2}} \operatorname{sech}[(2J)^{\frac{1}{2}}(X - X_0)], \quad X_0 = \pm(2J)^{-\frac{1}{2}} \tanh^{-1}\frac{L}{(2J)^{\frac{1}{2}}} \quad (J > \tfrac{1}{2}L^2), \quad (12a, b)$$

where the alternative signs in (12*b*) are vertically aligned with those in (11) and place the maximum of C at $X = X_0 \gtrless 0$ if $L > 0$. The parameter $\frac{1}{2}L^2$ typically exceeds $0.404A^2$ (e.g. $\frac{1}{2}L^2 = 11.7/32.6$ and $0.404A^2 = 0.35/0.36$ for the $n = 2/3$ mode in Barnard & Pritchard's tank), in consequence of which the trapping condition $J > \frac{1}{2}L^2$ typically requires $\lambda > 0$, i.e. $\sigma < \sigma_n$, which is the trapping condition in the absence of the wavemaker. (The approximations (10*a*, *b*) imply $\frac{1}{2}L^2 < 0.404A^2$ for $kd < 0.53$, but the deep-water approximation presumably fails for such small kd.)

Barnard & Pritchard (1972) obtain waves that appear to fall off exponentially from their wavemaker and, in at least one of the two cases for which they plot amplitude *vs.* distance (their figure 6), have profiles that are qualitatively similar to (12*a*), but these waves did not attain stationary states, for which reason quantitative comparisons with the present theory are not possible.

Wu, Keolian & Rudnick (1984) induce trapped waves at the form (2) with envelopes of the form (12) through vertical oscillation, at frequency 2σ, of a long channel. These parametrically excited waves are governed by a modified form of the cubic Schrödinger equation subject to null conditions at $X = \pm\infty$, and the agreement between theory (Miles 1984) and observation is reasonably close.

It is worth emphasizing that the solution described by (11) and (12) with the upper/lower choice of sign for $L \gtrless 0$ satisfies the boundary condition $\partial F/\partial X = 0$ at $X = X_0$. It follows that there is a single length (as opposed to an infinite, discrete set of lengths) for which the solution describes a standing wave in a tank of length aX_0/ϵ^2.

Barnard, Mahony & Pritchard (1977) consider the direct excitation of a cross-wave by an antisymmetric wavemaker oscillating near the cutoff frequency and obtain a steady-state equation that is similar to (4) if $\partial F/\partial \tau \equiv 0$ but with different coefficients and a boundary condition that differs significantly from (7). They give a solution (footnote following their (3.10), in which it appears that the 4 in the exponential should be a 2) that can be reduced to a hyperbolic secant and is a counterpart of (12).

3. Cnoidal waves

The trapped wave (12) is a limiting case of a one-parameter family of cnoidal waves for which the squared modulus m of the elliptic cosine may take values in $(0, \frac{1}{2})$ if $J < 0$ and in $(\frac{1}{2}, 1)$ if $J > 0$. The downstream boundary condition for $m < 1$ may be imposed at some finite value of X and determines an infinite, discrete sequence of resonance conditions (which are amplitude-dependent). The limit $m \uparrow 1$ with $J > 0$ yields the solution (12). The limit $m \downarrow 0$ with $J < 0$ yields a sinusoidal wave. The joint limit $J \to 0$ and $m \to \frac{1}{2}$ requires special attention, but does yield a cnoidal wave.

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